Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum dwell-time

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Abstract

Alternative formulations for stability analysis of switched systems under minimum dwell-time are proposed. As opposed to the hybrid conditions derived in [1], the obtained ones are affine in the system matrices and may be extended to uncertain switched systems with time-varying uncertainties. Additionally, the low number of decision variables allows us to losslessly derive convex robust stabilization conditions using a specific class of state-feedback control law. Several examples illustrate the approach.

Index Terms

Switched systems; dwell-time; stabilization; robustness; sum of squares

I. INTRODUCTION

Switched systems [1]–[9] are very flexible modeling tools appearing in several fields such as switching control laws [5], [10], networked control systems [11], electrical devices/circuits [12], [13], congestion modeling and control in networks [14]–[17], etc. When switching between a family of asymptotically stable subsystems holds in a way that is independent of the state of system, stability under minimum and average dwell-times have been shown to be relevant concepts of stability [2], [18] for which certain criteria have been proposed. Hybrid conditions, consisting of joint continuous-time and discrete-time conditions, for characterizing minimum dwell-time have been recently proposed in [1] where it is shown that the use of quadratic Lyapunov functions may lead to better results than previous ones. Even more importantly,

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homogeneous Lyapunov functions have been proved to be able to formulate nonconservative conditions for minimum dwell-time analysis [19], [20]. However, extending these important results to uncertain systems, time-varying systems and control design is quite difficult due to the presence of exponential terms that are not applicable to time-varying systems and would create strongly nonconvex terms in the design conditions.

Looped-functionals [21]–[23] have been shown to provide an alternative framework for dwell-time analysis of switched systems which remains compatible with uncertain switched systems, time-varying subsystems and, potentially, nonlinear switched systems. They also allow for the derivation of tractable conditions for robust stability analysis under mode-dependent dwell-time, a stability concept which permits instability of subsystems [23]. However, the structure of the conditions and the large number of decision variables makes the derivation of computationally attractive synthesis conditions an hardly possible task. The approach proposed in this paper aims at overcoming the computational drawbacks of looped-functionals related to control design and numerical complexity.

The contribution of the paper is manifold. First, alternative minimum dwell-time stability conditions, shown to be equivalent to those obtained in [1], are provided. The advantage of the proposed conditions lies in their affine dependence in the system matrices, permitting then their extension to uncertain systems with time-varying subsystems, as opposed to the conditions of [1] that are only applicable to time-invariant subsystems. The approach is then pushed further by providing, in a lossless way, convex stabilization conditions using a specific time-varying state-feedback control law. This provides a solution to the open problem of stabilizing a linear switched system with guaranteed minimum dwell-time¹. The price to pay for these interesting possibilities lies in the characterization of minimum dwell-time stability in terms of infinite-dimensional convex semidefinite programs, which may be hard to solve when the considered system is of large dimension. Sum of squares programming [24], [25] is employed to solve the resulting feasibility problems for some moderate size problems.

Outline: The structure of the paper is as follows: in Section II preliminary definitions and results are given. Section III is devoted to minimum dwell-time stability analysis while Section IV addresses minimum dwell-time stabilization. Examples are considered in the related sections.

¹See [1], page 1916, penultimate paragraph.

Notations: The sets of symmetric and positive definite matrices of dimension n are denoted by \mathbb{S}^n and $\mathbb{S}^n_{\succ 0}$ respectively. Given two symmetric real matrices A and B, the inequalities $A \succ (\succeq) B$ mean that A - B is positive (semi)definite. For a square real matrix A, the operator $\mathrm{Sym}(A)$ stands for the sum $A + A^\intercal$.

II. PRELIMINARIES

A. System definition

From now on, the following class of linear switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t)
x(t_0) = x_0$$
(1)

are considered where $x, x_0 \in \mathbb{R}^n$ are the state of the system and the initial condition, respectively. The switching signal σ is defined as a left-continuous piecewise constant function $\sigma: [0, \infty) \to \{1, \dots, N\}$. At some point, the matrices A_i of the subsystems will be uncertain and/or time-varying, this will be explicitly stated when this is the case. To avoid any Zeno behavior, we also assume that the sequence of switching instants $\{t_1, t_2, \dots\}$ is strictly increasing and does not admit any accumulation point.

B. Stability with periodic switching times

Let us start with a period switching stability result which allows us to state the main ideas in a simple context. By periodic switching, it is meant here that switching times are periodic, i.e. $t_{k+1} = t_k + \bar{T}$, for some $\bar{T} > 0$, but the sequence of subsystems may be not. Therefore, periodic systems theory does apply here. The following result will be shown to be directly involved in the derivation of the results on minimum dwell-time stability in the next section.

Theorem 1 (Stability with periodic switching times): The following statements are equivalent:

a) The quadratic form $V(x(t), \sigma(t)) = x(t)^{\mathsf{T}} P_{\sigma(t)} x(t)$, $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1, \dots, N$, is a discrete-time Lyapunov function for the switched system (1) with \bar{T} -periodic switching times in the sense that the inequality

$$V(x(t_{k+1}), \sigma(t_{k+1})) - V(x(t_k), \sigma(t_k)) \le -\mu ||x(t_k)||_2^2$$
(2)

holds for some $\mu > 0$, all $x(t_k) \in \mathbb{R}^n$ and all $k \in \mathbb{N}$.

b) There exist matrices $P_i \in \mathbb{S}_{\succ 0}^n$, i = 1, ..., N such that the LMIs

$$e^{A_i^{\mathsf{T}}\bar{T}}P_ie^{A_i\bar{T}} - P_i \prec 0 \tag{3}$$

hold for all $i, j = 1, \dots, N, i \neq j$.

c) There exist differentiable **indefinite** matrix functions $R_i:[0,\bar{T}]\mapsto\mathbb{R}^n$, $R_i(0)\in\mathbb{S}^n_{\succ 0}$, $i=1,\ldots,N$, and a scalar $\varepsilon>0$ such that the LMIs

$$A_i^{\mathsf{T}} R_i(\tau) + R_i(\tau) A_i - \dot{R}_i(\tau) \prec 0 \tag{4}$$

and

$$R_i(\bar{T}) - R_j(0) + \varepsilon I \le 0 \tag{5}$$

hold for all $\tau \in [0, \bar{T}]$ and all $i, j = 1, \dots, N, i \neq j$.

d) There exist differentiable **indefinite** matrix functions $S_i:[0,\bar{T}]\mapsto\mathbb{S}^n,\ S_i(\bar{T})\in\mathbb{S}^n_{\succ 0},$ $i=1,\ldots,N,$ and a scalar $\varepsilon>0$ such that the LMIs

$$A_i^{\mathsf{T}} S_i(\tau) + S_i(\tau) A_i + \dot{S}_i(\tau) \leq 0 \tag{6}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \le 0 \tag{7}$$

hold for all $\tau \in [0, \bar{T}]$ and all $i, j = 1, \dots, N, i \neq j$.

Proof: Proof of a) \Leftrightarrow b): Assume $\sigma(t_k) = j$ and $\sigma(t_k + \tau) = i$, $\tau \in (0, \bar{T}]$. Then, we have

$$V(x(t_{k+1}), \sigma(t_{k+1})) - V(x(t_k), \sigma(t_k)) = x(t_k)^{\mathsf{T}} \left[e^{A_i^{\mathsf{T}} \bar{T}} P_i e^{A_i \bar{T}} - P_j \right] x(t_k)$$
(8)

and there exists $\mu > 0$ such that (2) holds if and only if (3) holds. The proof is complete.

Proof of c) \Rightarrow **b):** Assume c) holds. Solving (4) for $R_i(\tau)$ yields

$$R_i(\tau) \succeq e^{A_i^{\mathsf{T}}\tau} R_i(0) e^{A_i \tau} \tag{9}$$

and thus

$$e^{A_i^{\mathsf{T}}\bar{T}}R_i(0)e^{A_i\bar{T}} - R_i(\bar{T}) \leq 0. \tag{10}$$

From (5), we have that $R_i(\bar{T}) \leq R_j(0) - \varepsilon I$ and therefore, combining this with (10), we obtain

$$e^{A_i^{\mathsf{T}}\bar{T}}R_i(0)e^{A_i\bar{T}} - R_i(0) + \varepsilon I \leq 0 \tag{11}$$

which implies in turn that (3) holds with $P_i = R_i(0)$.

Proof of d) \Rightarrow **b):** The proof follows the same lines as the one above and is omitted. Note that, in this case, (3) holds with $P_i = S_i(\bar{T})$.

Proof of b) \Rightarrow **c):** The proof is structured as follows: first, we prove that (4) admits solutions regardless of the stability of the system, showing that this condition can be assumed to be satisfied without loss of generality. The second part of the proof consists of combining statement b) with the solution set of (4) to prove that (5) holds.

Assume (3) holds with $P_i = R_i(0)$ and some $Y_{ij} \succ 0$ as

$$e^{A_i^{\mathsf{T}}\bar{T}}R_i(0)e^{A_i\bar{T}} - R_j(0) = -Y_{ij}, i, j = 1, \dots, N, \ i \neq j.$$
(12)

The set of all solutions $R_i(\tau)$ to (4) can be defined as the set of solutions to the matrix equality

$$A_i^{\mathsf{T}} R_i(\tau) + R_i(\tau) A_i - \dot{R}_i(\tau) = -Z_i(\tau), \ Z_i(\tau) \succeq 0$$

$$\tag{13}$$

where Z_i is arbitrary. All the solutions are therefore parametrized as

$$R_{i}(\tau) = e^{A_{i}^{\mathsf{T}}\tau} R_{i}(0) e^{A_{i}\tau} + \int_{0}^{\tau} e^{A_{i}^{\mathsf{T}}(\tau - s)} Z_{i}(s) e^{A_{i}(\tau - s)} ds$$
(14)

where $R_i(0) > 0$ verifies (12). Therefore, we have proved that (13) can be considered as fulfilled, independently of the stability of the system, which concludes the first part of the proof.

The second part of the proof simply consists of setting $\tau = \bar{T}$ in (14) to obtain

$$e^{A_i^{\mathsf{T}}\bar{T}}R_i(0)e^{A_i\bar{T}} - R_i(\bar{T}) = -\tilde{Z}_i(\bar{T})$$
 (15)

where $\tilde{Z}_i(\bar{T}) = \int_0^{\bar{T}} e^{A_i^{\mathsf{T}}(\bar{T}-s)} Z_i(s) e^{A_i(\bar{T}-s)} ds \succeq 0$. Then subtracting (15) from (12), we get

$$-R_{i}(0) + R_{i}(\bar{T}) = -Y_{ij} + \tilde{Z}_{i}(\bar{T}). \tag{16}$$

Since, the $Z_i(s)$'s, and therefore the $\tilde{Z}_i(\bar{T})$'s, can be chosen as small as desired and the positive definite Y_{ij} 's are arbitrary, this then implies that (5) holds. The proof is complete.

Proof of c) \Leftrightarrow **d):** Assume d) holds for some $S_i(\tau)$, it is immediate to see that $R_i(\tau) := S_i(\bar{T} - \tau)$ solves (4) and (5). Reverting the argument proves the equivalence.

The advantages of the conditions of statements c) and d) over conditions of statement b) are several. First of all, the conditions are convex in the system matrices A_i , allowing then for an immediate extension to the uncertain case. Secondly, the low number of decision matrices tends to suggest the possibility of deriving tractable synthesis conditions. Finally, conditions of statements c) and d) are more appealing from a computational perspective than the conditions

obtained from looped-functionals [23] since the number of decision variables and constraints is smaller.

A peculiarity of the approach is that, by virtue of (9), one has to only impose $R_i(0)$ to be positive definite to obtain a positive definite $R(\tau)$ for $\tau \in [0, \bar{T}]$. The same remark holds for $S_i(\tau)$ and $S_i(\bar{T})$. This reduces the number of constraints that have to be considered.

The compensation for these interesting properties is the consideration of infinite-dimensional feasibility problems which are hard to solve. Polynomial programming techniques [26] developed recently, such as sum of squares programming [24], however provide an adapted framework for solving such problems by restricting the matrix functions $R(\tau)$ and $S(\tau)$ to polynomial matrix functions. The package SOSTOOLS [25] together with the semidefinite programming solver SeDuMi [27] supply the necessary material for solving such problems.

III. STABILITY OF SWITCHED SYSTEMS UNDER MINIMUM DWELL-TIME

In this section, a minimum dwell-time stability result is recalled first. Then, new formulations for stability under minimum dwell-time are provided and extended to the uncertain case. In what follows, we shall consider the family of sequence of switching times

$$\mathbb{I}_{\eta} := \{ \{ t_1, t_2, \dots \} : T_k := t_{k+1} - t_k \in [\eta, +\infty), \ k \in \mathbb{N} \}, \ \eta > 0$$
 (17)

which contains sequences satisfying the minimum dwell-time η .

A. A preliminary result

The following result consist of a reformulation of the minimum dwell-time stability result of [1], but reproved according to some ideas taken from [21]–[23].

Lemma 2 (Minimum dwell-time): The following statements are equivalent:

a) The quadratic form $V(x(t), \sigma(t)) = x(t)^{\mathsf{T}} P_{\sigma(t)} x(t)$, $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1, \dots, N$, is a Lyapunov function for the system (1) in the sense that

$$\dot{V}(x(t),i) \le -\mu ||x(t)||_2^2, \ t \in (t_k, t_{k+1})$$
(18)

and

$$V(x(t_{k+1}), \sigma(t_{k+1})) - V_j(x(t_k), \sigma(t_k)) \le -\zeta ||x(t_k)||_2^2$$
(19)

hold for some $\mu, \zeta > 0$, all $x(t), x(t_k) \in \mathbb{R}^n$ and any sequence $\{t_k\}_{k \in \mathbb{N}} \in \mathbb{I}_{\bar{T}}$.

b) There exist $P_i \in \mathbb{S}_{>0}^n$, i = 1, ..., N, such that the LMIs

$$A_i^{\mathsf{T}} P_i + P_i A_i \prec 0 \tag{20}$$

and

$$e^{A_i^{\mathsf{T}}\theta}P_ie^{A_i\theta} - P_j \prec 0 \tag{21}$$

hold for all i, j = 1, ..., N, $i \neq j$ and all $\theta \geq \bar{T}$.

c) There exist $P_i \in \mathbb{S}_{>0}^n$, i = 1, ..., N, such that the LMIs

$$A_i^{\mathsf{T}} P_i + P_i A_i \prec 0 \tag{22}$$

and

$$e^{A_i^{\mathsf{T}}\bar{T}}P_ie^{A_i\bar{T}} - P_i \prec 0 \tag{23}$$

hold for all $i, j = 1, ..., N, i \neq j$.

Moreover, when one of the above statements holds, the switched system (1) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\bar{T}}$.

Proof: **Proof of a)** \Leftrightarrow **b):** Assume first that (18) holds. This then implies that

$$|x(t)|^{\mathsf{T}}[A_iP_i + P_iA_i]x(t) \le -\mu||x(t)||_2^2$$

for all $x(t) \in \mathbb{R}^n$, $t \neq t_k$, $k \in \mathbb{N}$, which is equivalent to stating that (20) holds. The proof that (19) implies (21) follows the same lines. Reverting the arguments proves that b) \Rightarrow a).

Proof of b) \Rightarrow **c)**: Immediate.

Proof of c) \Rightarrow **b):** Let us consider that (22) and (23) hold. A Taylor expansion of

$$\mathcal{L}_i(\theta) := e^{A_i^{\mathsf{T}} \theta} P_i e^{A_i \theta}$$

around $\theta = \theta_0 \ge \bar{T}$ yields

$$\mathcal{L}_i(\theta_0 + \delta) := e^{A_i^{\mathsf{T}}\theta_0} P_i e^{A_i\theta_0} + \delta e^{A_i^{\mathsf{T}}\theta_0} \operatorname{Sym}[A_i^{\mathsf{T}} P_i] e^{A_i\theta_0} + o(\delta)$$
(24)

where $o(\cdot)$ is the Landau small-o notation. Hence, we have

$$\mathcal{L}_{i}(\theta_{0} + \delta) - \mathcal{L}_{i}(\theta_{0}) = \delta e^{A_{i}^{\mathsf{T}}\theta_{0}} \operatorname{Sym}[A_{i}^{\mathsf{T}}P_{i}]e^{A_{i}\theta_{0}} + o(\delta). \tag{25}$$

Since (22) holds, the right-hand side is negative definite for all $\theta_0 \geq \bar{T}$, therefore we have

$$e^{A_i^{\mathsf{T}}(\bar{T}+\delta)} P_i e^{A_i(\bar{T}+\delta)} \leq e^{A_i^{\mathsf{T}}\bar{T}} P_i e^{A_i\bar{T}}$$
(26)

for all $\delta \geq 0$ and thus

$$e^{A_i^{\mathsf{T}}\theta}P_ie^{A_i\theta} - P_j \leq e^{A_i^{\mathsf{T}}\bar{T}}P_ie^{A_i\bar{T}} - P_j \prec 0$$
 (27)

holds for all $\theta \geq \bar{T}$. The proof is complete.

B. Nominal stability under minimum dwell-time

The theorem below addresses the case of minimum dwell-time stability for systems without uncertainties:

Theorem 3 (Minimum Dwell-Time): The following statements are equivalent:

a) There exist matrices $P_i \in \mathbb{S}^n_{\succ 0}$, i = 1, ..., N, such that the LMIs

$$A_i^{\mathsf{T}} P_i + P_i A_i \prec 0 \tag{28}$$

and

$$e^{A_i^{\mathsf{T}}\bar{T}}P_ie^{A_i\bar{T}} - P_i \prec 0 \tag{29}$$

hold for all $i, j = 1, ..., N, i \neq j$.

b) There exist matrix functions $R_i:[0,\bar{T}]\mapsto\mathbb{S}^n,\ i=1,\ldots,N,\ R_i(0)\in\mathbb{S}^n_{\succ 0}$, and a scalar $\varepsilon>0$ such that the LMIs

$$A_i^{\mathsf{T}} R_i(0) + R_i(0) A_i < 0 \tag{30}$$

$$A_i^{\mathsf{T}} R_i(\tau) + R_i(\tau) A_i - \dot{R}_i(\tau) \leq 0 \tag{31}$$

and

$$-R_j(0) + R_i(\bar{T}) + \varepsilon I \le 0 \tag{32}$$

hold for all $\tau \in [0, \bar{T}]$ and all $i, j = 1, \dots, N, i \neq j$.

c) There exist matrix functions $S_i:[0,\bar{T}]\mapsto\mathbb{S}^n,\ i=1,\ldots,N,\ S_i(\bar{T})\in\mathbb{S}^n_{\succ 0}$, and a scalar $\varepsilon>0$ such that the LMIs

$$A_i^{\mathsf{T}} S_i(\bar{T}) + S_i(\bar{T}) A_i \prec 0 \tag{33}$$

$$A_i^{\mathsf{T}} S_i(\tau) + S_i(\tau) A_i + \dot{S}_i(\tau) \leq 0 \tag{34}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \le 0 \tag{35}$$

hold for all $\tau \in [0, \bar{T}]$ and all $i, j = 1, \dots, N, i \neq j$.

Moreover, when one of the above statements holds, the switched system (1) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\bar{T}}$.

C. Robust stability under minimum dwell-time

Let us assume now that the system is uncertain and the (possibly time-varying) matrices A_i belong to the following polytopes

$$A_i \in \mathcal{A}_i := \operatorname{co}\left\{A_i^{[1]}, \dots, A_i^{[M]}\right\}$$
(36)

for some M>0 and all $i=1,\ldots,N.$ Before stating the main results, it is important to define the set Φ_i^θ as

$$\mathbf{\Phi}_i^{\theta} := \{ \Phi_i(\theta) : \Phi_i(s) \text{ solves (38)}, \ \lambda(s) \in \Lambda_N, \ s \in [0, \theta] \}$$
(37)

where Λ_M is the M-unit simplex and

$$\frac{d\Phi_i(s)}{ds} = \left(\sum_{j=1}^M \lambda_j(s) A_i^{[j]}\right) \Phi_i(s), \ \Phi_i(0) = I, \ \lambda(s) \in \Lambda_M, \ s \ge 0.$$
 (38)

The complexity of the set Φ_i^{θ} emphasizes the difficulty of characterizing uncertain sets in the discrete-time framework. The proposed framework, however, allows us to circumvent this problem and yields the following theorem:

Theorem 4 (Robust Minimum Dwell-Time): The following statements are equivalent:

a) There exist matrices $P_i \in \mathbb{S}_{>0}^n$, i = 1, ..., N, such that the LMIs

$$A_i^{\mathsf{T}} P_i + P_i A_i \prec 0 \tag{39}$$

and

$$\Phi_i(\bar{T})^{\dagger} P_i \Phi_i(\bar{T}) - P_j \prec 0 \tag{40}$$

hold for all $\Phi_i(\bar{T}) \in \mathbf{\Phi}_i^{\bar{T}}$ and all $i, j = 1, \dots, N$, $i \neq j$.

b) There exist matrix functions $R_i:[0,\bar{T}]\mapsto\mathbb{S}^n,\ R_i(0)\in\mathbb{S}^n_{\succ 0},\ i=1,\ldots,N,$ and a scalar $\varepsilon>0$ such that the LMIs

$$\left(A_i^{[k]}\right)^{\mathsf{T}} R_i(0) + R_i(0) A_i^{[k]} \prec 0 \tag{41}$$

$$\left(A_i^{[k]}\right)^{\mathsf{T}} R_i(\tau) + R_i(\tau) A_i^{[k]} - \dot{R}_i(\tau) \leq 0$$
 (42)

and

$$-R_j(0) + R_i(\bar{T}) + \varepsilon I \le 0 \tag{43}$$

hold for all $\tau \in [0, \overline{T}]$, all $i, j = 1, \dots, N$, $i \neq j$ and all $k = 1, \dots, M$.

c) There exist matrix functions $S_i:[0,\bar{T}]\mapsto\mathbb{S}^n,\ S_i(\bar{T})\in\mathbb{S}^n_{\succ 0},\ i=1,\ldots,N,$ and a scalar $\varepsilon>0$ such that the LMIs

$$\left(A_i^{[k]}\right)^{\mathsf{T}} S_i(\bar{T}) + S_i(\bar{T}) A_i^{[k]} \prec 0 \tag{44}$$

$$\left(A_{i}^{[k]}\right)^{\mathsf{T}} S_{i}(\tau) + S_{i}(\tau)A_{i}^{[k]} + \dot{S}_{i}(\tau) \leq 0 \tag{45}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \le 0 \tag{46}$$

hold for all $\tau \in [0, \overline{T}]$, all $i, j = 1, \dots, N$, $i \neq j$ and all $k = 1, \dots, M$.

Moreover, when one of the above statements holds, the uncertain switched system (1)-(36) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\bar{T}}$.

Proof: The proof follows the same lines as the one of Theorem 3 and exploit very standard arguments on the convexity of the stability conditions and the convexity of the polytopes A_i . It is thus omitted.

D. Examples

Illustrative examples are given here. The conditions of Theorem 3 are enforced using sum-of-squares programming [24], [28] and the semidefinite programming solver SeDuMi [27]. Thus, in the examples below, the matrix functions R_i 's or S_i 's will be considered as polynomials of chosen degree to be determined.

Example 5: Let us consider the system (1) with matrices [1]

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix}. \tag{47}$$

Using the initial result on minimum-dwell-time of [18], the upper-bound 6.66 on the minimum dwell-time is found. For comparison, the average dwell-time condition of [2] yields the conservative value 16.5554. Using the minimum dwell-time result of [1], i.e. statement b) of Theorem

	order of R_i	System (47)	System (48)	System (49)	
Theorem 3, c)	2	3.6769	0.6796	2.0302	
	4	2.9281	0.6226	1.9193	
	6	2.9048	0.6222	1.9167	
Theorem 3, b)	_	2.7508	0.6222	1.9134	

TABLE I

Upper bounds on the minimal dwell-time of Systems (47), (48) and (49) determined using Theorem 3 for different degrees for the polynomial functions R_i .

3, the upper bound 2.7508 is obtained. This justifies the use of the result of [1] for dwell-time analysis of linear switched systems. Statement c) or d) of Theorem 3 yields the minimum dwell-time estimates summarized in Table I. We can see that the proposed method allows to approach the upper-bound on the minimum dwell-time as the degree of the R_i 's increases.

Example 6: Let us consider the system (1) with matrices [29]

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -9 & -1 \end{bmatrix}.$$
 (48)

Using Theorem 3, b), the upper bound value 0.6222 on the minimal dwell-time is found. Using then Theorem 3, c), we obtain the upper bounds of Table I. We can see that the upper-bound determined using Theorem 3, b) can be actually retrieved with Theorem 3, c) when the polynomials R_i are of degree 6.

Example 7: Let us consider the system (1) with matrices [29]

$$A_{1} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ -2 & 1 & -1 \end{bmatrix}, \quad A_{2} = \begin{bmatrix} -1 & 0 & 6 \\ -2 & -1 & -5 \\ 0 & 3 & -1 \end{bmatrix}.$$
 (49)

Using Theorem 3, b), the minimal dwell-time upper bound value 1.9134 is found. Using then Theorem 3, c), we obtain the upper bounds of Table I. We can see that by choosing polynomials R_i 's of order 6, the result of Theorem 3, b) is almost retrieved.

Example 8: Let us consider the system (48), but assume now that the matrices A_1 and A_2

	order of R_i	$\kappa = 0.1$	$\kappa = 0.3$	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	$\kappa = 1.1$	$\kappa = 1.3$
Theorem 4, c)	2	0.6833	0.8035	0.9697	1.1888	1.4929	1.9698	2.8789
	4	0.6792	0.7485	0.8115	0.9122	1.1277	1.5062	2.4590
	6	0.6784	0.7411	0.7972	0.8769	1.0037	1.1977	1.9374
Theorem 4, b)	_	0.6759	0.7298	0.7689	0.8128	0.8673	0.9512	1.1475

TABLE II

Upper bounds on the minimal dwell-time of system (48)-(50) determined using statement b) of Theorem 4 (constant uncertainties) and c) of Theorem 4 (time-varying uncertainties) for different orders for R_i

are now time-varying and belong to the following polytopes:

$$\mathcal{A}_{1} = \operatorname{co} \left\{ \begin{bmatrix} 0 & 1 \\ -2 - \kappa & -1 \\ 0 & 1 \\ -9 - \kappa & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 + \kappa & -1 \\ 0 & 1 \\ -9 + \kappa & -1 \end{bmatrix} \right\},$$
(50)

for some $\kappa > 0$ representing the amplitude of the perturbation. We then obtain the results of Table II. Note, however, that while the results obtained with statement b) of Theorem 4 are only valid for time-invariant subsystems, those obtained with statement c) also apply to systems with time-varying uncertainties. Comparing the results directly should be therefore done with care.

IV. STABILIZATION OF SWITCHED SYSTEMS WITH GUARANTEED MINIMUM DWELL-TIME

Stabilization using state-feedback is considered in this section. Robust stabilization is omitted since it straightforwardly follows from nominal stabilization and robust stability analysis. To derive our nominal stabilization result, let us redefine system (1) as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \tag{51}$$

where $u_i \in \mathbb{R}^{m_i \times n}$, i = 1, ..., N are the control inputs. We further assume that the control law is given by

$$u_{\sigma(t)}(t) = K_{\sigma(t)}(t)x(t)$$
(52)

where

$$K_{\sigma(t_k+\tau)}(t_k+\tau) = \begin{cases} \tilde{K}_{\sigma(t_k)}(\tau) & \text{if } \tau \in [0,\bar{T}) \\ \tilde{K}_{\sigma(t_k)}(\bar{T}) & \text{if } \tau \in [\bar{T},T_k) \end{cases}$$

$$(53)$$

where the functions $\tilde{K}_i:[0,\bar{T}]\to\mathbb{R}^{m_i\times n},\ i=1,\ldots,N$ have to be determined such that the closed-loop system (51)-(52)-(53) is asymptotically stable with prescribed minimum dwell-time \bar{T} .

Theorem 9 (Stabilization with minimum Dwell-Time): The following statements are equivalent:

a) There exist matrices $P_i \in \mathbb{S}_{>0}^n$, i = 1, ..., N such that the LMIs

$$(A_i + B_i \tilde{K}_i(\bar{T})) P_i + P_i (A_i + B_i \tilde{K}_i(\bar{T}))^{\dagger} \prec 0$$
(54)

and

$$\Psi_i(\bar{T})P_i\Psi_i(\bar{T})^{\mathsf{T}} - P_i \prec 0 \tag{55}$$

hold for all $i, j = 1, ..., N, i \neq j$ where

$$\frac{d\Psi_i(s)}{ds} = (A_i + B_i K_i(s)) \,\Psi_i(s), \ \Psi_i(0) = I, \ s \ge 0.$$
 (56)

b) There exist matrix functions $S_i:[0,\bar{T}]\mapsto\mathbb{S}^n,\ S_i(\bar{T})\in\mathbb{S}^n_{\succ 0},\ U_i:[0,\bar{T}]\mapsto\mathbb{R}^{m_i\times n},\ i=1,\ldots,N$ and a scalar $\varepsilon>0$ such that the LMIs

$$Sym[A_iS_i(\bar{T}) + B_iU_i(\bar{T})] < 0$$
(57)

$$Sym[A_iS_i(\tau) + B_iU_i(\tau)] + \dot{S}_i(\tau) \le 0$$
(58)

and

$$S_i(0) - S_i(\bar{T}) + \varepsilon I \le 0 \tag{59}$$

hold for all $\tau \in [0, \overline{T}]$ and all $i, j = 1, \dots, N, i \neq j$.

Moreover, when one of the above statements holds, the closed-loop system (51)-(52)-(53) is asymptotically stable with minimum dwell-time \bar{T} and suitable matrix functions \tilde{K}_i are given by

$$\tilde{K}_i(\tau) = U_i(\tau)S_i(\tau)^{-1}.$$
(60)

Proof: The goal is then to show that statement a) is a necessary and sufficient condition for the existence a stabilizing state-feedback of the form (52)-(53) for system (51), in the sense of Theorem 3. The closed-loop system is given by

$$\dot{x}(t) = (A_i + B_i K_i (t - t_k)) x(t), \ t \in [t_k, t_{k+1}).$$
(61)

The key idea is to use Lemma 2 and Theorem 3 to prove stability of the closed-loop system. To simplify the derivation of convex synthesis conditions, the adjoint system with reverse-time of (61) given by

$$\dot{y}(t) = (A_i + B_i K_i (t - t_k))^{\mathsf{T}} y(t), \ t \in (t_k, t_{k+1}]$$
(62)

is considered. The crucial point here is that proving stability of (62) is equivalent to proving stability of (61). Noting first that for all $\theta \geq \bar{T}$, $\tilde{K}_i(\theta) = \tilde{K}_i(\bar{T})$ and following the same arguments as in the proof of Lemma 2, we find that condition (22) exactly becomes condition (54). In the same way, condition (23) equivalently becomes condition (55), which proves exactness of statement a).

Equivalence with statement b) follows from Theorem 3 and from the change of variables $U_i(\tau) = \tilde{K}_i(\tau)P_i(\tau)$. The proof is complete.

Example 10: Let us consider the system (51) with matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -5 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}, \ B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
 (63)

Setting $\bar{T}=0.1$ and choosing the polynomials S and Y as first-order polynomials, we obtain the control gains

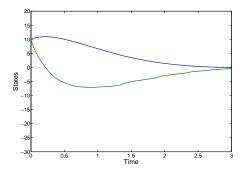
$$K_{1}(\tau) = \frac{1}{d_{1}(\tau)} \left[0.9251 + 15.4274\tau + 1.5713\tau^{2} - 3.7623 + 7.1348\tau + 1.2093\tau^{2} \right]$$

$$K_{2}(\tau) = \frac{1}{d_{2}(\tau)} \left[-2.8369 + 1.6128\tau + 0.4961\tau^{2} - 15.4803 - 21.3893\tau - 4.2782\tau^{2} \right]$$
(64)

where $d_1(\tau) = 0.6915 + 4.2476\tau + 1.9162\tau^2$ and $d_2(\tau) = 2.1420 + 3.7553\tau + 0.9021\tau^2$. The state and control-gain trajectories are depicted in Fig. 1 and 2 where we can see that the obtained controller stabilizes the switched system correctly. Note that it is also possible to identify phases where the controller maintains its value to $\tilde{K}_i(\bar{T})$.

V. CONCLUSION

New conditions for characterizing minimum dwell-times for uncertain linear switched systems with time-varying uncertainties have been provided. Thanks to the structural properties of the stability criterion, convex state-feedback design conditions have been derived, providing then a solution to the open-problem of the stabilization of linear switched systems with guaranteed



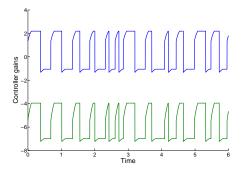


Fig. 1. State trajectories of the closed-loop system (51)-(63).

Fig. 2. Control gain trajectories of the closed-loop system system (51)-(63).

dwell-time. The flexibility of the framework makes it easily adaptable to other types of dwell-times results such as stability under mode-dependent dwell-time, as proposed in [23]. Possible extensions include the use of homogeneous Lyapunov functions as in [20] and the consideration of nonlinear switched systems with polynomial vector field.

REFERENCES

- [1] J. C. Geromel and P. Colaneri, "Stability and stabilization of continuous-time switched linear systems," *SIAM Journal on Control and Optimization*, vol. 45(5), pp. 1915–1930, 2006.
- [2] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in 38th Conference on Decision and Control, Phoenix, Arizona, USA, 1999, pp. 2655–2660.
- [3] J. P. Hespanha, "Extending Lasalle's invariance principle to switched linear systems," in *Decision and Control, Orlando, Florida, USA*, vol. 3, 2001, pp. 2496–2501.
- [4] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach," *IEEE Transactions on Automatic Control*, vol. 47(11), pp. 1883–1887, 2002.
- [5] D. Liberzon, Switching in Systems and Control. Birkhäuser, 2003.
- [6] J. Zhao and D. J. Hill, "Dissipativity theory for switched systems," *IEEE Transactions on Automatic Control*, vol. 53(4), pp. 941–953, 2008.
- [7] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems Part ii: (pre)asymptotically stable compact sets," *IEEE Transactions on Automatic Control*, vol. 53(3), pp. 734–748, 2008.
- [8] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54(2), pp. 308–322, 2009.
- [9] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29(2), pp. 28–93, 2009.

- [10] B. Demirel, C. Briat, and M. Johansson, "Supervisory control design for networked systems with time-varying communication delays," in *4th IFAC Conference on analysis and design of hybrid systems*, Eindovhen, the Netherlands, 2012, pp. 133–140.
- [11] M. C. F. Donkers, L. Hetel, W. P. M. H. Heemels, N. van de Wouw, and M. Steinbuch, "Stability analysis of networked control systems using a switched linear systems approach," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, R. Majumdar and P. Tabuada, Eds. Springer Berlin / Heidelberg, 2009, vol. 5469, pp. 150–164.
- [12] S. Almér, H. Jujioka, U. T. Jönsson, C. Y. Kao, D. Patino, P. Riedinger, T. Geyer, A. Beccuti, G. Papafotiou, M. Morari, A. Wernrud, and A. Rantzer, "Hybrid control techniques for switched-mode DC-DC converter part I: The step-down topology," in *American Control Conference, New York, USA*, 2007, pp. 5450–5457.
- [13] A. G. Beccuti, G. Papafotiou, M. Morari, S. Almér, H. Fujioka, U. T. Jönsson, C. Y. Kao, A. Wernrud, A. Rantzer, M. Bâja, H. Cormerais, and J. Buisson, "Hybrid control techniques for switched-mode DC-DC converters part II: The step-up topology," in *American Control Conference, New York, USA*, 2007, pp. 5464–5471.
- [14] J. Hespanha, S. Bohacek, K. Obraczka, and J. Lee, "Hybrid modeling of TCP congestion control," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, M. Di Benedetto and A. Sangiovanni-Vincentelli, Eds. Springer Berlin / Heidelberg, 2001, vol. 2034, pp. 291–304.
- [15] R. Shorten, F. Wirth, and D. Leith, "A positive systems model of TCP-like congestion control: asymptotic results," *IEEE Transactions on Networking*, vol. 14(3), pp. 616–629, 2006.
- [16] C. Briat, H. Hjalmarsson, K. H. Johansson, G. Karlsson, U. T. Jönsson, and H. Sandberg, "Nonlinear state-dependent delay modeling and stability analysis of internet congestion control," in 49th IEEE Conference on Decision and Control, Atlanta, USA, 2010, pp. 1484–1491.
- [17] C. Briat, H. Hjalmarsson, K. H. Johansson, G. Karlsson, U. T. Jönsson, H. Sandberg, and E. A. Yavuz, "An axiomatic fluid-flow model for congestion control analysis," in 50th IEEE Conference on Decision and Control, Orlando, Florida, USA, 2011, pp. 3122–3129.
- [18] A. S. Morse, "Supervisory control of families of linear set-point controllers part 1: Exact matching," *IEEE Transactions on Automatic Control*, vol. 41(10), pp. 1413–1431, 1996.
- [19] F. Wirth, "A converse Lyapunov theorem for linear parameter-varying and linear switching systems," *SIAM Journal on Control and Optimization*, vol. 44(1), pp. 210–239, 2005.
- [20] G. Chesi, P. Colaneri, J. C. Geromel, R. Middleton, and R. Shorten, "A nonconservative lmi condition for stability of switched systems with guaranteed dwell-time," *IEEE Transactions on Automatic Control*, vol. 57(5), pp. 1297–1302, 2012.
- [21] C. Briat and A. Seuret, "A looped-functional approach for robust stability analysis of linear impulsive systems," *Systems & Control Letters*, vol. 61, pp. 980–988, 2012.
- [22] —, "Convex dwell-time characterizations for uncertain linear impulsive systems," *IEEE Transactions on Automatic Control*, vol. 57(12), pp. 3241–3246, 2012.
- [23] —, "Affine minimal and mode-dependent dwell-time characterization for uncertain switched linear systems," to appear in IEEE Transactions on Automatic Control (May 2013), 2013.
- [24] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, Pasadena, California, 2000.
- [25] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, SOSTOOLS: Sum of squares optimization toolbox for MATLAB, Available from http://www.cds.caltech.edu/sostools and http://www.mit.

- edu/~parrilo/sostools, 2004.
- [26] G. Chesi, "LMI techniques for optimization over polynomials in control: A survey," *IEEE Transactions on Automatic Control*, vol. 55(11), pp. 2500–2510, 2010.
- [27] J. F. Sturm, "Using SEDUMI 1.02, a Matlab Toolbox for Optimization Over Symmetric Cones," *Optimization Methods and Software*, vol. 11, no. 12, pp. 625–653, 2001.
- [28] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous polynomial forms for robustness analysis of uncertain systems*, ser. Lecture Notes in Control and Information Sciences. Berlin Heidelberg: Springer-Verlag, 2009.
- [29] G. Chesi, P. Colaneri, J. C. Geromel, R. Middleton, and R. Shorten, "Computing upper-bounds of the minimum dwell time of linear switched systems via homogeneous polynomial Lyapunov functions," in *American Control Conference*, Baltimore, Maryland, USA, 2010, pp. 2487–2492.